

METABELIAN THIN BEAUVILLE p -GROUPS

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ABSTRACT. A non-cyclic finite p -group G is said to be *thin* if every normal subgroup of G lies between two consecutive terms of the lower central series and $|\gamma_i(G) : \gamma_{i+1}(G)| \leq p^2$ for all $i \geq 1$. In this paper, we determine Beauville structures in metabelian thin p -groups.

1. INTRODUCTION

Beauville surfaces [2, page 159] are complex surfaces of general type constructed from two orientable regular hypermaps of genus at least 2, with the same automorphism groups. A finite group which arises as an automorphism group linked to a Beauville surface is called a *Beauville group*.

A group-theoretical formulation of Beauville groups can be given as follows. Given a finite group G and a couple of elements $x, y \in G$, we define

$$\Sigma(x, y) = \bigcup_{g \in G} (\langle x \rangle^g \cup \langle y \rangle^g \cup \langle xy \rangle^g),$$

that is, the union of all subgroups of G which are conjugate to $\langle x \rangle$, to $\langle y \rangle$ or to $\langle xy \rangle$. Then G is a Beauville group if and only if the following conditions hold:

- (i) G is a 2-generator group.
- (ii) There exists a pair of generating sets $\{x_1, y_1\}$ and $\{x_2, y_2\}$ of G such that $\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = 1$.

Then $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are said to form a *Beauville structure* for G .

By using this characterization, one can study whether a given finite group is a Beauville group. For example, Catanese [5] showed that a finite abelian group is a Beauville group if and only if it is isomorphic to $C_n \times C_n$, with $n > 1$ and $\gcd(n, 6) = 1$. On the other hand, a remarkable result, proved independently by Guralnick and Malle [11] and by Fairbairn, Magaard and Parker [6] in 2012, is that every non-abelian finite simple group other than A_5 is a Beauville group.

If p is a prime, not much was known about Beauville p -groups until very recently (see [1] and [3]). In [7, Theorem 2.5], Fernández-Alcober and Gül extended Catanese's criterion in the case of p -groups from abelian groups to a much wider family of groups, including all p -groups having a 'good behaviour' with respect to taking powers, and in particular groups of class $< p$.

A non-cyclic finite p -group G is said to be *thin* if every normal subgroup of G lies between two consecutive terms of the lower central series and $|\gamma_i(G) : \gamma_{i+1}(G)| \leq p^2$ for all $i \geq 1$. Indeed, these groups are 2-generator. Thus it is natural to ask whether they are Beauville or not. Furthermore, the study of thin p -groups is also motivated by the fact that they usually give examples of groups whose power structures are not so well-behaved. Well-known examples of thin p -groups are p -groups of maximal class and quotients of the Nottingham group. In [7], all Beauville quotients of the Nottingham group were determined. Thanks to the ill-behaved power structure, the first explicit infinite family of Beauville 3-groups was given by considering quotients

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of the Nottingham group over \mathbb{F}_3 . In [8], the existence of Beauville structures in the most important families of groups of maximal class, in particular in metabelian p -groups of maximal class, was studied.

The goal of this paper is to complete the study of Beauville structures in metabelian thin p -groups. In this paper, we will exclude p -groups of maximal class from our consideration of thin groups. This means, in particular, that the prime 2 is excluded [12, Theorem III.11.9]. Then according to Theorem A in [4], if G is a metabelian thin p -group, then $\text{cl}(G) \leq p + 1$. If the class is $< p$, then the existence of Beauville structures can be determined by using Corollary 2.6 in [7]. Thus we restrict to groups of class p or $p + 1$. The main result of this paper is as follows.

Theorem A. *Let G be a metabelian thin p -group of class p or $p + 1$ for $p \geq 5$. Then there are four cases in which there is a Beauville structure:*

- (i) $\text{cl}(G) = p$ and $|\gamma_p(G)| = p^2$.
- (ii) $\text{cl}(G) = p + 1$.
- (iii) $\text{cl}(G) = p$, $|\gamma_p(G)| = p$ and $G^p = \gamma_{p-1}(G)$.
- (iv) $\text{cl}(G) = p$, $|\gamma_p(G)| = p$, $G^p = \gamma_p(G)$ and G has at least three maximal subgroups of exponent p .

Notation. We use standard notation in group theory. If G is a group, $H, K \leq G$ and $N \trianglelefteq G$, then $H \equiv K \pmod{N}$ means $HN/N = KN/N$. If p is a prime, then we write G^{p^i} for the subgroup generated by all powers g^{p^i} as g runs over G , and $\Omega_i(G)$ for the subgroup generated by the elements of G of order at most p^i . The exponent of G , denoted by $\exp G$, is the maximum of the orders of all elements of G .

2. PROOF OF THE MAIN RESULT

In this section, we give the proof of Theorem A. We begin by giving some properties of metabelian thin p -groups. Firstly, we recall the following more general result of Meier-Wunderli. If G is a metabelian 2-generator p -group, then

$$(1) \quad G^p \geq \gamma_p(G)$$

(see [13], Theorem 3).

Observe that the only thin abelian p -group is the elementary abelian group of order p^2 and we refer to its lattice of normal subgroups as a *diamond*. It then follows that if G is thin, then G/G' is elementary abelian of order p^2 , and hence G is 2-generator. Also, the lower and upper central series of a thin p -group coincide [4, Corollary 2.2].

Note that if G is a thin p -group and $g \in \gamma_i(G) \setminus \gamma_{i+1}(G)$, then

$$(2) \quad \gamma_{i+1}(G) = [g, G]\gamma_{i+2}(G)$$

(see [4], Lemma 2.1).

Now let G be a metabelian thin p -group. By Theorem A in [4], we have the following:

- (i) $\gamma_{p+1}(G)$ is cyclic and $\gamma_{p+2}(G) = 1$.
- (ii) The lattice of normal subgroups of G consists of a diamond on top, followed by a chain of length 1, at most $p - 2$ diamonds, plus possibly another chain of length 1.

As a consequence, $\text{cl}(G) \leq p + 1$, and $|G| \leq p^{2p}$. We next recall the power structure of a metabelian thin p -group.

Lemma 2.1. *Let G be a metabelian thin p -group, and l be the largest integer such that $G^p \leq \gamma_l(G)$. Then $3 \leq l \leq p$, $\gamma_{l+1}(G)$ is cyclic, $\gamma_{l+2}(G) = 1$ and $\gamma_2(G)^p \leq \gamma_{l+1}(G)$.*

Proof. See the proofs of Lemmas 1.2, 1.3 and 3.3 in [4]. \square

The next corollary follows directly from Lemma 2.1.

Corollary 2.2. *Let G be a metabelian thin p -group. Then $|G^p| \leq p^3$.*

Lemma 2.3. *Let G be a metabelian thin p -group such that its lattice of normal subgroups ends with a chain. Then the order of G^p cannot be p^2 .*

Proof. If G is of class $p+1$, then $G^p = \gamma_p(G)$, and hence $|G^p| = p^3$. Thus we assume that $\text{cl}(G) = c \leq p$. Next observe that if M is a maximal subgroup of G , then $\text{cl}(M) < c \leq p$, and so M is regular.

Now suppose, on the contrary, that $|G^p| = p^2$. Consider the quotient group $\overline{G} = G/\gamma_c(G)$, which is regular. Then $|\overline{G}^p| = p$, and hence $|\overline{G} : \Omega_1(\overline{G})| = p$. Write $\Omega_1(\overline{G}) = M/\gamma_c(G)$ for some maximal subgroup M of G . Since \overline{G} is regular, $\exp \Omega_1(\overline{G}) = p$, and hence $M^p \leq \gamma_c(G)$. This implies that $|M^p| = |M : \Omega_1(M)| \leq p$ as M is regular. It then follows that $G' \leq \Omega_1(M)$, and thus $\exp G' = p$.

On the other hand, if M is an arbitrary maximal subgroup of G , we have $G' \leq M$ and since $\exp G' = p$, we get $G' \leq \Omega_1(M) \leq M$. Then $|M^p| = |M : \Omega_1(M)| \leq p$. Since G is thin, this implies that $M^p \leq \gamma_c(G)$ for any maximal subgroup M . But $G^p = \langle M^p \mid M \text{ maximal in } G \rangle \leq \gamma_c(G)$. Thus $|G^p| \leq p$, which is a contradiction. \square

We finally recall a commutator relation between the generators of G . More specifically, if G is a metabelian thin p -group, then to every $x \in G \setminus G'$ there corresponds a y such that $G = \langle x, y \rangle$ and

$$(3) \quad [y, x, x, x] \equiv [y, x, y, y]^h \pmod{\gamma_5(G)}.$$

where h is a quadratic non-residue modulo p [4, Theorem B].

Before we proceed to prove Theorem A, we will determine which metabelian thin 3-groups are Beauville groups.

Theorem 2.4. *A metabelian thin 3-group is a Beauville group if and only if it is one of $\text{SmallGroup}(3^5, 3)$, $\text{SmallGroup}(3^6, 34)$, or $\text{SmallGroup}(3^6, 37)$.*

Proof. Let G be a metabelian thin 3-group. Then $|G| \leq 3^6$. Note that the smallest Beauville 3-group $S = \text{SmallGroup}(3^5, 3)$ is of order 3^5 , and it is the only Beauville 3-group of that order [1]. Furthermore, by using the computer algebra system GAP [10], it can be seen that S is metabelian thin. Thus, if $|G| = 3^5$, then G is a Beauville group if and only if $G \cong S$. We next assume that $|G| = 3^6$. It has been shown in [1] that there are only three Beauville 3-groups of order 3^6 , namely $S = \text{SmallGroup}(3^6, n)$ for $n = 34, 37, 40$. However, if $n = 40$ then S is not thin since $|Z(S)| = 9$ and thus $Z(S) \neq \gamma_4(S)$. On the other hand, if $n = 34$ or 37 then again by using the computer algebra system GAP, one can show that S is a metabelian thin 3-group. Consequently, G is a Beauville group if and only if $G \cong S$ for $n = 34$ or 37 . \square

Thus we assume that $p \geq 5$. Let G be a metabelian thin p -group with $\text{cl}(G) = p$ or $p+1$. Then we have three cases: G is of class $p+1$, or is of class p and $|\gamma_p(G)| = p^2$, or is of class p and $|\gamma_p(G)| = p$. In the first two cases, we have $G^p \leq \gamma_p(G)$. It then follows from (1) that $G^p = \gamma_p(G)$. Note that we also have $\gamma_2(G)^p \leq \gamma_{p+1}(G)$, by Lemma 2.1. On the other hand, in the last case if l is the largest integer satisfying $G^p \leq \gamma_l(G)$, then $l = p-1$ or p and hence $\gamma_p(G) \leq G^p \leq \gamma_{p-1}(G)$.

Our first step is to calculate the p th powers of $x^t y$ modulo $\gamma_{p+1}(G)$ for all $0 \leq t \leq p-1$ if $G = \langle x, y \rangle$ and $\gamma_2(G)^p \leq \gamma_{p+1}(G)$. To this purpose, we need the following lemma.

Lemma 2.5. [14, Lemma 6] *Let G be a metabelian p -group and $x, y \in G$. Set $\sigma_1 = y$ and $\sigma_i = [\sigma_{i-1}, x]$ for $i \geq 2$. Then*

$$(xy)^p = x^p y^p \sigma_2^{\binom{p}{2}} \dots \sigma_p^{\binom{p}{p}} z,$$

where

$$z = \prod_{i=1}^{p-1} \prod_{j=1}^{p-1} [\sigma_{i+1,j} \sigma_1]^{C(i,j)},$$

and

$$C(i, j) = \sum_{k=1}^{p-1} \binom{k}{i} \binom{k}{j}.$$

Lemma 2.6. *Let G be a metabelian thin p -group such that $\gamma_2(G)^p \leq \gamma_{p+1}(G)$. If x and y are the generators of G satisfying (3), then for all $0 \leq t \leq p-1$*

$$(4) \quad (x^t y)^p \equiv (x^p)^t y^p [y, x, p-2 y]^{\frac{-2t}{1-ht^2}} [y, x, p-3 y, x]^{\frac{2t^2}{1-ht^2}} \pmod{\gamma_{p+1}(G)}.$$

Proof. By Lemma 2.5, we have

$$(x^t y)^p = (x^p)^t y^p [y, x^t]^{\binom{p}{2}} \dots [y, p-1 x^t]^{\binom{p}{p}} \prod_{i=1}^{p-1} \prod_{j=1}^{p-1} [y, x^t, y]^{\binom{p}{p}} [y, x^t, y]^{\binom{p}{p}}.$$

Since $\gamma_2(G)^p \leq \gamma_{p+1}(G)$, it then follows that

$$(x^t y)^p \equiv (x^p)^t y^p [y, p-1 x^t] \prod_{\substack{1 \leq i, j \\ i+j \leq p-1}} [y, x^t, y]^{\binom{p}{p}} \pmod{\gamma_{p+1}(G)}.$$

Note that for $i+j > 0$, $C(i, j)$ is the coefficient of $u^i v^j$ in $\sum_{k=0}^{p-1} (1+u)^k (1+v)^k$, where

$$\sum_{k=0}^{p-1} (1+u)^k (1+v)^k \equiv ((u+v) + uv)^{p-1} \pmod{p}.$$

In the previous expression the monomials of total degree less than p appear only in $(u+v)^{p-1} \equiv \sum_{r=0}^{p-1} (-1)^r u^r v^{p-r-1} \pmod{p}$, and hence

$$C(i, j) \equiv \begin{cases} 0 \pmod{p} & \text{if } i+j < p-1, \\ (-1)^i \pmod{p} & \text{if } i+j = p-1. \end{cases}$$

Thus the condition $\gamma_2(G)^p \leq \gamma_{p+1}(G)$ implies that

$$(x^t y)^p \equiv (x^p)^t y^p \prod_{i=1}^{p-1} [y, x^t, p-i-1 y]^{(-1)^i} \pmod{\gamma_{p+1}(G)}.$$

On the other hand, notice that for $1 \leq t \leq p-1$

$$(5) \quad [y, x^t, x^t, x^t] \equiv [y, x^t, y, y]^{ht^2} \pmod{\gamma_5(G)}.$$

Since G is metabelian, for every $a, b \in G$ and $c \in G'$ we have $[c, a, b] = [c, b, a]$, and this, together with (5), yields that

$$[y, x^t, p-i-1 y]^{(-1)^i} \equiv \begin{cases} [y, x^t, p-2 y]^{-(ht^2)^{s-1}} \pmod{\gamma_{p+1}(G)} & \text{if } i = 2s-1, \\ [y, x^t, p-3 y, x^t]^{(ht^2)^{s-1}} \pmod{\gamma_{p+1}(G)} & \text{if } i = 2s, \end{cases}$$

and hence

$$(x^t y)^p \equiv (x^p)^t y^p \left([y, x, x_{p-2} y]^{-t} [y, x, x_{p-3} y, x]^{t^2} \right)^{\sum_{s=1}^{(p-1)/2} (ht^2)^{s-1}} \pmod{\gamma_{p+1}(G)}.$$

Note that since h is a quadratic non-residue, we have $\sum_{s=1}^{(p-1)/2} (ht^2)^{s-1} = \frac{2}{1-ht^2}$. Consequently, we get

$$(x^t y)^p \equiv (x^p)^t y^p [y, x, x_{p-2} y]^{\frac{-2t}{1-ht^2}} [y, x, x_{p-3} y, x]^{\frac{2t^2}{1-ht^2}} \pmod{\gamma_{p+1}(G)},$$

for $0 \leq t \leq p-1$, as desired. \square

Lemma 2.7. *Let G be a metabelian thin p -group such that $|\gamma_p(G)| \geq p^2$ and let x and y be the generators of G satisfying (3). Then for every $t_0 \in \{0, 1, \dots, p-1\}$ there exist at most three distinct $t \in \{0, 1, \dots, p-1\}$ such that*

$$(6) \quad \langle (x^{t_0} y)^p \rangle \equiv \langle (x^t y)^p \rangle \pmod{\gamma_{p+1}(G)}.$$

Proof. Since $|\gamma_p(G)| \geq p^2$, we have $G^p = \gamma_p(G)$ and $\gamma_2(G)^p \leq \gamma_{p+1}(G)$, by Lemma 2.1. Also note that $|\gamma_p(G) : \gamma_{p+1}(G)| = p^2$. Notice that, as a consequence of (2), $l = [y, x, x_{p-2} y]$ and $m = [y, x, x_{p-3} y, x]$ are linearly independent modulo $\gamma_{p+1}(G)$. Thus (l, m) is a basis of $\gamma_p(G)$ modulo $\gamma_{p+1}(G)$. If we set $x^p \equiv l^\alpha m^\beta \pmod{\gamma_{p+1}(G)}$ and $y^p \equiv l^\gamma m^\delta \pmod{\gamma_{p+1}(G)}$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{F}_p$, then by (4), we have

$$(7) \quad (x^t y)^p \equiv l^{\gamma + \alpha t - \frac{2t}{1-ht^2}} m^{\delta + \beta t + \frac{2t^2}{1-ht^2}} \pmod{\gamma_{p+1}(G)}.$$

Observe that as rational functions in t , neither $f(t) = \gamma + \alpha t - \frac{2t}{1-ht^2}$ nor $g(t) = \delta + \beta t + \frac{2t^2}{1-ht^2}$ are zero.

We now fix $t_0 \in \{0, 1, \dots, p-1\}$. Then (6) holds if and only if there exists $\lambda \in \mathbb{F}_p^*$ such that

$$f(t) = \lambda f(t_0) \quad \text{and} \quad g(t) = \lambda g(t_0).$$

If $f(t_0) = 0$ or $g(t_0) = 0$, then we have $f(t) = 0$ or $g(t) = 0$, that is $(1 - ht^2)(\gamma + \alpha t) - 2t = 0$ or $(1 - ht^2)(\delta + \beta t) + 2t^2 = 0$. Otherwise, we have $\frac{f(t)}{f(t_0)} = \frac{g(t)}{g(t_0)}$. Then $g(t_0)f(t) - f(t_0)g(t) = 0$, that is

$$g(t_0)((1 - ht^2)(\gamma + \alpha t) - 2t) - f(t_0)((1 - ht^2)(\delta + \beta t) + 2t^2) = 0,$$

which is a polynomial in t of degree ≤ 3 . Thus in every case, there are at most three distinct $t \in \{0, 1, \dots, p-1\}$ such that $\langle (x^{t_0} y)^p \rangle \equiv \langle (x^t y)^p \rangle \pmod{\gamma_{p+1}(G)}$. \square

Lemma 2.8. *Let G be a metabelian thin p -group such that $\gamma_2(G)^p \leq \gamma_{p+1}(G)$. If M is a maximal subgroup of G and $a, b \in M \setminus G'$, then $\langle a \rangle^p \equiv \langle b \rangle^p \pmod{\gamma_{p+1}(G)}$.*

Proof. If we write $b = a^i c$ for some $c \in G'$ and for some integer i not divisible by p , then by the Hall-Petrescu collection formula (see [12, III.9.4]), we have

$$(a^i c)^p = a^{pi} c^p c_2^{\binom{p}{2}} c_3^{\binom{p}{3}} \dots c_p,$$

where $c_j \in \gamma_j(\langle a, c \rangle) \leq \gamma_{j+1}(G)$. Thus $(a^i c)^p \equiv a^{pi} \pmod{\gamma_{p+1}(G)}$, and hence $\langle a \rangle^p \equiv \langle b \rangle^p \pmod{\gamma_{p+1}(G)}$. \square

Remark 2.9. If we replace x with x^* , where $x^* \in G \setminus G'$ is not a power of x , there exists a corresponding y^* satisfying (3). Then $x \in \langle (x^*)^{t_0} y^*, G' \rangle \setminus G'$ for some $0 \leq t_0 \leq p-1$, and according to Lemma 2.8, we have $\langle x^p \rangle \equiv \langle ((x^*)^{t_0} y^*)^p \rangle \pmod{\gamma_{p+1}(G)}$. It then follows from Lemma 2.7 that there exist at most three distinct $t \in \{0, 1, \dots, p-1\}$ such that $\langle x^p \rangle \equiv \langle (x^t y)^p \rangle \pmod{\gamma_{p+1}(G)}$.

The following corollary is an immediate consequence of Lemmas 2.7 and 2.8.

Corollary 2.10. *Let G be a metabelian thin p -group such that $|\gamma_p(G)| \geq p^2$. If M is a maximal subgroup of G , then there exist at most two maximal subgroups M_1, M_2 different from M such that $M^p \equiv M_1^p \equiv M_2^p \pmod{\gamma_{p+1}(G)}$.*

Before we present the main result, we also need the following remark.

Remark 2.11. Let G be a finite 2-generator p -group. Then we can always find elements $x, y \in G \setminus \Phi(G)$ such that x, y and xy fall into the given three maximal subgroups of G . Let M_1, M_2 and M_3 be three maximal subgroups of G . Choose $x \in M_1 \setminus \Phi(G)$ and $y \in M_2 \setminus \Phi(G)$. Since each element in the set $\{xy^j \mid 1 \leq j \leq p-1\}$ falls into different maximal subgroups, there exists $1 \leq j \leq p-1$ such that $xy^j \in M_3 \setminus \Phi(G)$. Thus if we put $x^* = x$ and $y^* = y^j$, then elements in the triple $\{x^*, y^*, x^*y^*\}$ fall into the given three maximal subgroups.

We are now ready to prove Theorem A. We deal separately with the cases in the theorem.

Theorem 2.12. *Let G be a metabelian thin p -group with $\text{cl}(G) = p$ such that $|\gamma_p(G)| = p^2$, where $p \geq 5$. Then G has a Beauville structure in which one of the two triples has all elements of order p^2 .*

Proof. We divide our proof into three cases depending on the number of maximal subgroups whose p th powers coincide, and in every case, we take into account Corollary 2.10 and Remark 2.11. First of all, note that since G has at most three maximal subgroups of exponent p and $p \geq 5$, there are at least three maximal subgroups of exponent p^2 .

Case 1: Assume that there is a 1-1 correspondence between maximal subgroups M_i of exponent p^2 and M_i^p . Choose a set of generators $\{x_1, y_1\}$ such that $o(x_1) = o(y_1) = o(x_1y_1) = p^2$.

Case 2: Assume that there exist three maximal subgroups of exponent p^2 such that their p th power subgroups coincide. Then choose a set of generators $\{x_1, y_1\}$ such that x_1, y_1 and x_1y_1 fall into those maximal subgroups.

In both Case 1 and 2, since $p \geq 5$, we can choose another set of generators $\{x_2, y_2\}$ so that each pair of elements in $\{x_i, y_i, x_iy_i \mid i = 1, 2\}$ is linearly independent modulo G' by Remark 2.11.

Case 3: Assume that we are not in the first two cases. Then there exist two maximal subgroups M_1, M_2 of exponent p^2 such that $M_1^p = M_2^p$ and $M^p \neq M_1^p$ for all other maximal subgroups M .

Let us first deal with $p \geq 7$. We start by choosing a set of generators $\{x_1, y_1\}$ where $x_1 \in M_1$ and $y_1 \in M_2$ are such that $o(x_1y_1) = p^2$, say $x_1y_1 \in M_3$. Then there might be a maximal subgroup M_4 such that $M_3^p = M_4^p$ (note that there is no other $i \neq 3, 4$ satisfying $M_i^p = M_3^p$, otherwise we are in Case 2). Since $p \geq 7$, we can choose another set of generators $\{x_2, y_2\}$ so that $x_2, y_2, x_2y_2 \notin M_4$ and each pair of elements in $\{x_i, y_i, x_iy_i \mid i = 1, 2\}$ is linearly independent modulo G' , by Remark 2.11.

If $p = 5$ then by using the construction of metabelian thin p -groups in [4] and the computer algebra system GAP, one can show that there is no metabelian thin 5-group of class 5 such that $|\gamma_5(G)| = 5^2$ and 5th powers of maximal subgroups coincide in pairs. Thus in Case 3, there exists a maximal subgroup, say M_3 , of exponent 5^2 , where all other M^5 are different from M_3^5 . Then choose sets of generators $\{x_1, y_1\}$ and $\{x_2, y_2\}$ so that $x_1 \in M_1, y_1 \in M_2$ and $x_1y_1 \in M_3$ and each pair of elements in $\{x_i, y_i, x_iy_i \mid i = 1, 2\}$ is linearly independent modulo G' .

We claim that, in every case, $\{x_1, y_1\}$ and $\{x_2, y_2\}$ form a Beauville structure for G . If $A = \{x_1, y_1, x_1y_1\}$ and $B = \{x_2, y_2, x_2y_2\}$, then we need to show that

$$(8) \quad \langle a^g \rangle \cap \langle b^h \rangle = 1,$$

for all $a \in A$, $b \in B$ and $g, h \in G$. Note that $o(a) = p^2$ for every $a \in A$. Assume first that $o(b) = p$. If $\langle a^g \rangle \cap \langle b^h \rangle \neq 1$ for some $g, h \in G$, then $\langle b^h \rangle \subseteq \langle a^g \rangle$, and hence $\langle aG' \rangle = \langle bG' \rangle$, which is a contradiction, since a and b are linearly independent modulo G' . Thus we assume that $o(b) = p^2$. If (8) does not hold, then $\langle (a^g)^p \rangle = \langle (b^h)^p \rangle$, which contradicts the choice of b . \square

In order to deal with the case $\text{cl}(G) = p + 1$, we need the following lemma.

Lemma 2.13. [9, Lemma 4.2] *Let G be a finite group and let $\{x_1, y_1\}$ and $\{x_2, y_2\}$ be two sets of generators of G . Assume that, for a given $N \trianglelefteq G$, the following hold:*

- (i) $\{x_1N, y_1N\}$ and $\{x_2N, y_2N\}$ is a Beauville structure for G/N ,
- (ii) $o(u) = o(uN)$ for every $u \in \{x_1, y_1, x_1y_1\}$.

Then $\{x_1, y_1\}$ and $\{x_2, y_2\}$ is a Beauville structure for G .

Theorem 2.14. *Let G be a metabelian thin p -group with $\text{cl}(G) = p + 1$, where $p \geq 5$. Then G has a Beauville structure.*

Proof. By Theorem 2.12, $\overline{G} = G/\gamma_{p+1}(G)$ has a Beauville structure in which one of the two triples has all elements of order p^2 , i.e. they have the same order in both G and \overline{G} . Then we can apply Lemma 2.13 and thus G is a Beauville group. \square

We next analyze the case $\text{cl}(G) = p$ and $|\gamma_p(G)| = p$. Recall that we have $\gamma_p(G) \leq G^p \leq \gamma_{p-1}(G)$, and thus there are two possibilities:

- (i) $G^p = \gamma_{p-1}(G)$,
- (ii) $G^p = \gamma_p(G)$.

Observe that by Lemma 2.3, G^p cannot be a proper subgroup of $\gamma_{p-1}(G)$ of order p^2 .

Theorem 2.15. *Let G be a group in case (i). Then G has a Beauville structure.*

Proof. First of all, notice that there exists a pair of generators a and b of G such that a^p and b^p are linearly independent modulo $\gamma_p(G)$. By the Hall-Petrescu formula, we have

$$(a^t b)^p = a^{tp} b^p c_2^{\binom{p}{2}} \dots c_p^{\binom{p}{p}},$$

where $c_j \in \gamma_j(\langle a^t, b \rangle)$. Since $\gamma_2(G)^p \leq \gamma_p(G)$, by Lemma 2.1, we get

$$(a^t b)^p \equiv a^{tp} b^p \pmod{\gamma_p(G)}$$

for $1 \leq t \leq p-1$. Observe that, similarly to Lemma 2.8, for every maximal subgroup M , $m \in M$ and $c \in G'$, we have $(mc)^p \equiv m^p \pmod{\gamma_p(G)}$. It then follows that the power subgroups M^p are all different modulo $\gamma_p(G)$.

On the other hand, since $\overline{G} = G/\gamma_p(G)$ is of class $p-1$, it is a regular p -group such that $|\overline{G}^p| = p^2$. According to Corollary 2.6 in [7], \overline{G} is a Beauville group since $p \geq 5$. From the observation above, all elements outside G' are of order p^2 in both G and \overline{G} . Then we can apply Lemma 2.13 to conclude that G is a Beauville group. \square

Theorem 2.16. *Let G be a group in case (ii). Then G has a Beauville structure if and only if it has at least three maximal subgroups of exponent p .*

Proof. If the number of maximal subgroups of exponent p is less than three, then $\Omega_1(G)$ is contained in the union of at most two maximal subgroups. Since $|G^p| = p$, it then follows from Proposition 2.4 in [7] that G has no Beauville structure.

On the other hand, if at least three maximal subgroups have exponent p , then we choose a triple in which all elements have order p . Since $p \geq 5$, we can choose another triple such that each pair of elements in the union of the two triples is linearly independent modulo G' . Then G has a Beauville structure. \square

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